JOURNAL OF APPROXIMATION THEORY 10, 206–217 (1974)

Jackson's Theorem on Complex Arcs

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Communicated by Oved Shisha

INTRODUCTION

If Γ_0 is the unit interval then the celebrated theorem of Jackson [3] tells us that for every continuous function, $f(z)$, there is a polynomial, $P(z)$, of degree $< n$ such that

$$|f(z) - P(z)| \leqslant A\omega_f(1/n) \qquad \text{throughout } \Gamma_0 . \tag{1}$$

Here ω_f is the modulus of continuity of f and A is an absolute constant.

If we think of Γ_0 as part of the complex plane rather than merely part of the real line, then a natural question arises. Does (1) remain true when Γ_0 is replaced by some other arc? We will say that a Jordan arc Γ has the Jackson property, or, more briefly, has J, if (1) remains true when Γ is written for Γ_0.

The question of just when Γ does have J is apparently not a simple one. It is not even obvious that anything but a line segment has J! Equally frustrating, on the other hand, is that it is not easy to point to any Jordan arc which *fails* to have J! Nevertheless, the truth is that some do and some don't and in this paper we find conditions for determining which is which. These conditions fall short of a necessary and sufficient criterion, however, and we can only conjecture as to what such might be. One attractive conjecture is that Γ has J if and only if Γ is C^1, but we are far from this result. What we do prove is that rectifiability (i.e., BV) is necessary for J, that $C^{1+\delta}$ is sufficient for J, and that C^{1-} (appropriately defined) is not sufficient for J.

A special case which, so far, has us baffled is that of the simple "corner" $[0, 1] \cup [0, i]$. This isn't C^1 and so our previous conjecture would imply that this corner doesn't have J. We cannot even prove this! Another special case of some interest is that of a circular arc. Here (1) becomes equivalent to a statement regarding approximation by trigonometric polynomials.

* Supported in part by AFOSR 69-1736.

Namely, given a continuous f there exist C_0, C_1, $-C_{n-1}$ such that

$$\left| f(x) - \sum_{k=0}^{n-1} C_k e^{ikx} \right| \leqslant C\omega_f(1/n) \quad \text{on} \quad [0, 2\pi - \delta]. \tag{2}$$

Here δ is any positive number dand C depends only on δ.

It had been known for some time that the exponentials e^{ikx} with $k \geqslant 0$ were sufficient for approximation on $[0, 2\pi - \delta]$, but (2) gives the quantitative version of this result. This statement (2), which is a corollary of our $C^{1+\delta}$ theorem, was originally conjectured by Feinerman and in fact it was this conjecture which led to our present general formulation. One further remark is pertinent. We point out, namely, that the quantity $1/n$ has the significance of being the absolute lowest possible order of magnitude. Thus there are no arcs for which (1) could hold with $1/n \log \log n$ replacing $1/n$, for example. The hunt for arcs with J is therefore the hunt for arcs along which polynomial approximation is, in a sense, optimally good. (A proof of this will be given below.)

Definitions and Preliminary Remarks

We now give some necessary definitions.

Definition 1. $\mathscr{S}$ is the class of all complex valued functions, f, defined on the whole plane and satisfying $|f(z) - f(z')| \leqslant |z - z'|$ for all z, z'.

Definition 2. If S is any compact plane set then

$$\rho_n(S) = \max_{f \in \mathscr{S}} \min_{\deg P < n} \max_{z \in S} |f(z) - P(z)|.$$

Definition 3. If S is any compact plane set then

$$\epsilon_n(S) = \max_{z_0, z_1, \ldots, z_n \in S} \min_{i \neq j} |z_i - z_j|.$$

Definition 4. C^{1-} is the class of all complex Jordan arcs parametrized by $z = Z(t)$, $t \in [0, 1]$ with $Z(t) \in C^1$ and $Z'(t) \neq 0$ except possibly at $t = 1$. Intrinsically, these are curves of finite length such that if any neighborhood of the right endpoint is deleted the curve becomes C^1.

Definition 5. If S is a plane set and $\epsilon > 0$ then S_ϵ is the set of all points within ϵ of S.

Some general remarks are now in order.

Remark 1. Γ has J if and only if $\rho_n(\Gamma) = O(1/n)$.

Remark 2. $\rho_n(S) \geqslant \frac{1}{2}\epsilon_n(S)$.

Remark 3. If Γ is any arc (not degenerating to a single point) then there is an $a > 0$ such that $\epsilon_n(\Gamma) \geqslant a/n$.

We include this last remark because, together with R2, it illustrates the "absolute lower bound" nature of $1/n$ that we referred to previously.

We turn to the proofs. To see R3 we rotate if necessary so that Γ is not a line segment in the real direction (clearly ϵ_n is not affected by a rotation). Now let $A = \min_\Gamma \operatorname{Im} z$, $B = \max_\Gamma \operatorname{Im} z$ and note that $A < B$. Next choose $z_0, z_1, ..., z_n$ so that $\operatorname{Im} z_k$ are equally spaced between $A = \operatorname{Im} z_0$ and $B = \operatorname{Im} z_n$. Clearly, then, for $i \neq j$,

$$| z_i - z_j | \geqslant | \operatorname{Im}(z_i - z_j)| = \frac{B-A}{n} | i - j | \geqslant \frac{B-A}{n}$$

and R3 follows.

R2 is a standard result [4] and is proved as follows. Let $z_0, z_1, ..., z_n$ be an extremizing set in D3. The map $P(z) \to (P(z_0), P(z_1), ..., P(z_n))$ is a linear map from a space of n dimensions (polynomials of degree $< n$) into a space of $n + 1$ dimensions and as such must be deficient. This means that there must exist $\alpha_0, \alpha_1, ..., \alpha_n$ such that $\alpha_0 P(z_0) + \cdots \alpha_n P(z_n) = 0$ for all such polynomials. Now define $f(z_0), f(z_1), ..., f(z_n)$ by $f(z_k) = (\epsilon/2)\, \overline{Sg\alpha_k}$ where $\epsilon = \epsilon_n(S)$. Since, for $i \neq j$, $|f(z_i) - f(z_j)| \leqslant \epsilon/2 + \epsilon/2 = \epsilon \leqslant |z_i - z_j|$, it follows that f can be extended to lie in $\mathscr{S}$ (for example, one point at a time, by transfinite induction). We assert that for no P of degree $< n$ can we have $|f - P| < \epsilon/2$ throughout S or even at all the points $z_0, z_1, z_2, ..., z_n$. Namely

$$|\epsilon/2 - Sg\alpha_k P(z_k)| < \epsilon/2 \Rightarrow 0 < \operatorname{Re} Sg\alpha_k P(z_k) \Rightarrow 0 < \operatorname{Re} \alpha_k P(z_k), \text{ and this}$$

holding for all k would contradict $\sum \alpha_k P(z_k) = 0$. Thus we have produced an $f \in \mathscr{S}$ which cannot be approximated within $\epsilon/2$ and R2 is proved.

Finally, we turn to R1. One half is obvious, for if $\rho_n(\Gamma) \neq O(1/n)$ then there exist $f \in \mathscr{S}$ which cannot be approximated within C/n. This of course is a violation of J since $\omega_f(1/n) \leqslant 1/n$ for any $f \in \mathscr{S}$.

On the other hand, suppose that $\rho_n(\Gamma) \leqslant A/n$. As before, by transfinite induction, we may, if n is sufficiently large, extend $f(z)$ to the whole plane without increasing its modulus of continuity. Next cover the plane by an equilateral triangular mesh with the sides of the triangles equal to $1/n$. Then define $g(z)$ to be equal to $f(z)$ at all the mesh points and to be linear (in z and $\bar{z}$) inside each triangle. Two things follow from this construction. First of all, if we denote by V_z the nearest mesh point to z, then we have

$$\begin{aligned} |f(z) - g(z)| &\leqslant |f(z) - f(V_z)| + |g(z) - g(V_z)| \\ &\leqslant \omega_f(1/n) + \omega_g(1/n) \leqslant 2\omega_f(1/n). \end{aligned} \tag{3}$$

Secondly, within any triangle $g(z)$ is linear and so its gradient is bounded by its difference quotient at the vertices. In other words, $|\operatorname{grad} g(z)| \leqslant n\omega_f(1/n)$. But this tells us that for any z and z' we have $|g(z) - g(z')| \leqslant n\omega_f(1/n) |z - z'|$, or that $g(z)/n\omega_f(1/n) \in \mathscr{S}$.

By D2 then we can find a $P(z)$ of degree $< n$ such that, throughout Γ,

$$\left| \frac{g(z)}{n\omega_f(1/n)} - p(z) \right| \leqslant \rho(\Gamma) \leqslant A/n.$$

Multiplying out and setting $P(z) = n\omega_f(1/n)\, p(z)$ gives

$$|g(z) - P(z)| \leqslant A\omega_f(1/n).$$

Combining this with (3) gives (1) with $A + 2$ written for A. R1 is proved.

Basic Theorems

THEOREM 1. *If Γ has infinite length then Γ does not have J.*

Proof. By R1 and R2 we need only prove that $\epsilon_n(\Gamma) \neq O(1/n)$. Indeed, we show more: given A, $\epsilon_n(\Gamma) \geqslant A/n$ for all sufficiently large n. Let $Z(t)$, $0 \leqslant t \leqslant 1$, be a parametrization of Γ.

Since the arc length is infinite we may pick N so that

$$\sum_{j=1}^{N} |Z(j/N) - Z[(j-1)/N]| \geqslant 2A.$$

It follows that either

$$\sum_k |Z(2k/N) - Z[(2k-1)/N]| \geqslant A$$

or

$$\sum_k |Z[(2k+1)/N] - Z(2k/N)| \geqslant A,$$

and without loss of generality we may assume the former.

If we now use the continuity of the *inverse* function, we are guaranteed that, for n sufficiently large,

$$|t - s| \geqslant 1/N \rightarrow |Z(t) - Z(s)| \geqslant A/n.$$

Now divide each line segment $[Z[(2k - 1/N)], Z(2k/N)]$ into m equal parts where m is the greatest integer in $n/A \,|\, Z(2k/N) - Z[(2k - 1/N)]|$. This guarantees that the pieces have length at least A/n. Each of the subdivision points, in turn, is a projection of some point on Γ so that these points are separated by at least A/n. Also any two such points coming from different k have this same separation because of the continuity cited above.

The total number of points we have produced, however, is

$$> \sum_k \frac{n}{A} \left| Z\left(\frac{2k}{N}\right) - Z\left(\frac{2k-1}{N}\right) \right| \geqslant \frac{n}{A}. \qquad A = n$$

and so, by D3, the proof is complete.

THEOREM 2. *There exists an arc Γ which is C^{1-} but which doesn't have J.*

LEMMA 1. *If S consists of the three lines segments* $[a, b]$, $w \cdot [a, b]$, $w^2 \cdot [a, b]$, *where* $0 < a < b$ *and* $w = e^{2\pi i/3}$, *then*

$$\rho_n(S) \geqslant \frac{b}{3n^{2/3}} \left(\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}\right)^n.$$

Proof. Choose $f(z) = |z|$ and observe that $f \in \mathcal{S}$. Now choose $p(z)$ of degree $< n$ to be its best polynomial approximant on S. By D2, then, throughout S, $|f(z) - p(z)| \leqslant \rho_n(S) = \rho$.

This means that on $[a, b]$ we have $|z - p(z)| \leqslant \rho$, $|z - p(wz)| \leqslant \rho$, $|z - p(w^2z)| \leqslant \rho$, and adding these three inequalities gives

$$\left| z - \frac{p(z) + p(wz) + p(w^2z)}{3} \right| \leqslant \rho \quad \text{on} \quad [a, b].$$

Here we have a bound for a polynomial on $[a, b]$ and this, by the Tehebychev estimate, yields a bound for it throughout $[0 \cdot b]$. The result is that, on $[0, b]$,

$$\left| z - \frac{p(z) + p(wz) + p(w^2z)}{3} \right| \leqslant \rho \left| T_n\left(\frac{b+a}{b-a}\right)\right| \leqslant \rho \left(\frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}\right)^n.$$

If we finally write $z = bt^{1/3}$, we obtain

$$|t^{1/3} - Q(t)| \leqslant \frac{\rho}{b} \left(\frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}\right)^n \quad \text{throughout} \quad [0, 1], \tag{4}$$

where $Q(t)$ is the *polynomial* $[p(bt^{1/3}) + p(wbt^{1/3}) + p(w^2bt^{1/3})/3b]$.

But lower bounds on approximations to t^α are well known [1]! Indeed, these assure us that for all $Q(t)$ of degree $< n$ we must have $|t^{1/3} - Q(t)| \geqslant 1/3n^{2/3}$ for some t in [0, 1].

Comparing this to (4) proves the lemma.

Proof of Theorem 2. The idea is a simple one. We make up Γ by stringing together smaller and smaller replicas of sets S described in our lemma. Since this Γ will contain all of these S we will have $\rho_n(\Gamma) \geqslant \rho_n(S)$ and the lemma will produce an adequate lower bound. The details follow.

For $k = 1, 2, 3,...$ we choose $a = 2^{-9k}$, $b = 2^{-k}$, form the set S as in the lemma and translate it by 2^{-k}. The resulting sets we call S_k and we observe that they are disjoint, have total length finite, and lie in a bounded region. We can obviously connect them all by an arc Γ which is C^{1-}. For this Γ we have $\rho_n(\Gamma) \geqslant \rho_n(S_k)$ and since ρ_n is translation invariant the lemma insures that

$$\rho_n(S_k) \geqslant \frac{2^{-k}}{3n^{2/3}} \left(\frac{2^{4k} - 1}{2^{4k} + 1}\right)^n.$$

Choose k so that $2^{4k-4} \leqslant n < 2^{4k}$ and conclude that

$$\rho_n(S_k) \geqslant \frac{1}{6n^{1/4} \cdot n^{2/3}} \left(\frac{2^{4k} - 1}{2^{4k} + 1}\right)^{2^{4k}} \geqslant 1/54n^{11/12}.$$

Hence $\rho_n(\Gamma) \neq O(1/n)$ and R1 completes the proof.

THEOREM 3. *If Γ is a Jordan arc in $C^{1+\delta}$ then Γ has J.*

To facilitate the proof we find it more convenient to work with closed Jordan curves rather than with arcs—and so we turn to the related closed Jordan curve.

We may, by a linear map, assume that the endpoints of Γ are -2 and $+2$. We then perform an "exterior" map by writing $z = w + 1/w$. This maps the complement of Γ in a schlicht manner onto the exterior of some closed Jordan curve which we call K and maps the "two sides of Γ" onto the curve K. (We note that smoothness properties of Γ are inherited by K.)

The point is that approximation on K by polynomials in w and $1/w$ yields ordinary polynomial approximation on Γ. For if $|S(w) - f(w + 1/w)| < \epsilon$ on K, then changing w into $1/w$ gives $|S(1/w) - f(w + 1/w)| < \epsilon$ on K and adding yields

$$\left| \frac{S(w) + S(1/w)}{2} - f(w + 1/w) \right| < \epsilon$$

on K. But if $S(w)$ is of degree $< n$ in w and $1/w$, then $[S(w) + S(1/w)]/2$

will be of degree $\leqslant n$ in $w - 1/w$. Furthermore the m.o.c. of $f(w + 1/w)$ as a function of w is bounded by a constant multiplied by its m.o.c. as a function of $z = w - 1/w$. Thus to prove Theorem 3 it suffices to prove the following.

THEOREM 4. *Let K be a closed Jordan curve of class $C^{1+\delta}$ surrounding the origin in the w plane. For any $f(w) \in \mathscr{S}$ there is an $S(w)$ of degree $< n$ in w and $1/w$ such that*

$$|f(w) - S(w)| \leqslant A/n \qquad \textit{along } K.$$

Proof. This divides into three steps. First, a conformal map is made of a neighborhood of K onto a neighborhood of C, the unit circle. Second, approximations are made on the unit circle by the standard Jackson theorem. Then, upon return to K, these polynomials become analytic functions and in the third step we utilize the theory of polynomial expansions of analytic functions.

LEMMA 2. *If $T(\xi)$ is of degree k in ξ and $1/\xi$ and if $|T(\xi)| \leqslant M$ on C (the unit circle), then, for any ϵ in $(0, 1)$, $|T(\xi)| \leqslant M(1-\epsilon)^{-k}$ and $|T'(\xi)| \leqslant Mk(1-\epsilon)^{-(k+1)}$ on C_ϵ.*

Proof. $\xi^k T(\xi)$ is analytic in $|\xi| \leqslant 1$ so that, in $1 - \epsilon \leqslant |\xi| \leqslant 1$, $(1-\epsilon)^k |T(\xi)| \leqslant |\xi^k T(\xi)| \leqslant M$. Again, $\xi^{-k}T(\xi)$ is analytic in $1 \leqslant |\xi| \leqslant \infty$ so that in $1 \leqslant |\xi| \leqslant 1 + \epsilon$, $(1-\epsilon)^k |T(\xi)| \leqslant (1+\epsilon)^{-k} |T(\xi)| \leqslant |\xi^{-k}T(\xi)| \leqslant M$. Combining these two proves the first half. For the second half, simply apply Bernsteins theorem [5] to obtain $|T'(\xi)| \leqslant Mk$ on C and then use the first result with $k + 1$ replacing k. [Note that if $\epsilon < \frac{1}{2}$ then we can replace $(1-\epsilon)^{-1}$ by $e^{2\epsilon}$ in the above bounds. This form will sometimes prove handy.)

Step 1. Let K be parametrized by $w = w(\xi)$ where ξ runs around C, the unit circle, so that $w \in C^1$, $w' \epsilon \operatorname{Lip}(\delta)$, and $w' \neq 0$. By Jackson's theorem on the circle then we can find $T(\xi)$ of degree k in ξ and $1/\xi$ such that along C

$$|w(\xi) - T(\xi)| \leqslant Ak^{-(1+\delta)} \tag{5}$$

and

$$|w'(\xi) - T'(\xi)| \leqslant Ak^{-\delta}.$$

For k large enough we can weaken this last to read

$$\alpha \leqslant |T'(\xi)| \leqslant \beta \qquad \text{where} \quad 0 < \alpha < \beta. \tag{6}$$

We now show that for some fixed positive a (independent of k) $T(\xi)$ is schlicht on $C_{a/k}$. Suppose otherwise that $T(\rho\xi) = T(\rho'\xi')$ with $1 = |\xi| =$

$|\xi'|$ and $|1-\rho| \leqslant a/k$ and $|1-\rho'| \leqslant a/k$. Then $|T(\xi) - T(\xi')| \leqslant |T(\xi) - T(\rho\xi)| + |T(\xi') - |T(\rho'\xi')| \leqslant (1-\rho) M_1 + (1-\rho') M_1$ where $M_1 = \max_{C_{a/k}} |T'(\xi)|$. By Lemma 2 and (6), however, we have $M_1 \leqslant \beta(1-a/k)^{-k} \leqslant \beta(1-a)^{-1}$ and the above estimate becomes

$$|T(\xi) - T(\xi')| \leqslant 2a(1-a)^{-1}\beta/k \leqslant 4\beta(a/k) \qquad (\text{assuming } a < \tfrac{1}{2}).$$

Hence by (5), for k large enough, we deduce that

$$|w(\xi) - w(\xi')| \leqslant (4\beta + 1)(a/k). \tag{7}$$

Recall now that the inverse function to w has a continuous derivative and conclude from (7) that for some γ we have $|\xi - \xi'| \leqslant \gamma(a/k)$. In turn, this easily gives

$$|\rho\xi - \rho'\xi'| \leqslant (2+\gamma)\, a/k. \tag{8}$$

Now, by the power series expansion, we have

$$T(\rho'\xi') = T(\rho\xi) + (\rho'\xi' - \rho\xi)\, T'(\rho\xi) + \tfrac{1}{2}(\rho'\xi' - \rho\xi)^2\zeta, \tag{9}$$

where $|\zeta| \leqslant M_2 = \max_{C_{a/k}} |T''(\xi)|$. Again we apply Lemma 2 and obtain

$$M_2 \leqslant \beta k(1-a/k)^{-(k+1)} \leqslant \beta k(1-a)^{-2} \leqslant 4\beta k, \tag{10}$$

and this combined with (9) and the fact that $T(\rho'\xi') = T(\rho\xi)$ gives

$$|\rho'\xi' - \rho\xi|\, |T'(\rho\xi)| \leqslant \tfrac{1}{2} |\rho'\xi' - \rho\xi|^2\, 4\beta k.$$

Cancelling and applying (8) gives

$$|T'(\rho\xi)| \leqslant (4 + 2\gamma)\, \beta a. \tag{11}$$

Once more we have

$$|T'(\xi)| \leqslant |T'(\rho\xi)| + (1-\rho)\, M_2$$

and an application of (10) and (11) gives

$$|T'(\xi)| \leqslant (8 + 2\gamma)\, \beta a. \tag{12}$$

For a small enough, (12) and (6) are in contradiction [e.g., we can choose $a = \alpha/(9+2\gamma)\beta$] and the proof is complete.

Next, we observe that, for small enough fixed $a > 0$ there is a $b > 0$ such that

$$T(C_{a/k}) \supseteq K_{b/k}\,. \tag{13}$$

Simply apply Rouche's theorem [6] to $T(\xi) - w$ in the disc Δ: $|\xi - \xi_0| \leq a/k$ where $|\xi_0| = 1$ and $|w - w(\xi_0)| \leq b/k$. We have

$$T(\xi) - w = T(\xi_0) - w + (\xi - \xi_0)\, T'(\xi_0) + R(\xi).$$

By (5), for large k, $|T(\xi_0) - w| \leq 2b/k$ and by (10) $|R(\xi)| \leq \frac{1}{2}(a/k)^2 M_2 \leq 2a^2\beta/k$; finally, (6) gives $|T'(\xi_0)| \geq \alpha$. The result follows, therefore, as soon as $\alpha(a/k) \geq 2a^2\beta/k + 2b/k$ or $(\alpha/2)a - \beta a^2 \geq b$. In fact, if $a \leq \alpha/4\beta$ we can choose $b = (\alpha/4)a$. This applies, for example, if $a = Ak^{-\delta}$ for large k and so tells us that for any A there is a B such that

$$T(C_{Bk} - (1 + \delta)) \supseteq K_{Ak} - (1 + \delta).$$

Step 2. Now let $f \in \mathscr{S}$ so that, for small enough λ, $\lambda f(w(\xi))$ is in $\mathscr{S}$ as a function of ξ. Applying Jackson's theorem [3] for the circle therefore gives a $U(\xi)$ of degree m in ξ and $1/\xi$ such that

$$|f(w(\xi)) - U(\xi)| \leq A/m, \qquad |U'(\xi)| \leq A \quad \text{on} \quad C.$$

Next form the function $U(T^{-1}(w))$ and observe by (13) that it is analytic on $K_{b/k}$. Furthermore, it is bounded there by $\max_{C_{a/k}} |U(\xi)|$ which by Lemma 2 gives

$$|U(T^{-1}(w))| \leq A(1 - (a/k))^{-m} \leq Ae^{2am/k} \qquad \text{throughout} \quad K_{b/k}.$$

Step 3. We now dip into the theory of polynomial expansions of analytic functions. What we need specifically is the following lemma.

LEMMA 3. *Let K be a closed Jordan curve of class $C^{1+\delta}$ surrounding the origin and let $f(w)$ be analytic and bounded by M in K_ϵ. There must exist an $S(w)$ of degree $< n$ in w and $1/w$ such that all along K $|f(w) - S(w)| \leq (AM/\epsilon^3)\, e^{-c\epsilon n}$ (A and c being constants depending only on K).*

Applying Cauchy's theorem gives

$$f(w) = \frac{1}{2\pi i}\int_{B_1} \frac{f(t)}{t - w}\, dt - \frac{1}{2\pi i}\int_{B_2} \frac{f(t)}{t - w}\, dt,$$

where B_1, B_2 are, respectively, the outer and inner boundaries of K_ϵ. This expresses $f(w)$ as the difference of two functions. The first is analytic and bounded by AM/ϵ throughout the $\epsilon/2$ neighborhood, say, of the "inside" of K, while the second is analytic and bounded by AM/ϵ throughout such a neighborhood of the "outside" of K. Thus our Lemma 3 emerges from two applications of the more classical looking lemma below.

LEMMA 4. *Let R be a bounded region whose boundary is a Jordan curve of class $C^{1+\delta}$. Let $f(w)$ be analytic and bounded by M in R_ϵ. There exists $p(w)$ of degree $< n$ (in w alone) such that, throughout R,*

$$|f(w) - p(w)| \leqslant (AM/\epsilon^2)\, e^{-c\epsilon n}.$$

(Here A and c are positive constants depending only on R.)

Proof. We use the method of Faber [7]. Namely, introduce the mapping function $\mu(\xi) = \lambda\xi + c_0 + c_1\,\xi^{-1} + c_2\,\xi^{-2} + \cdots$, $\lambda > 0$, which maps the exterior of the unit disc onto the exterior of R. Because R is bounded by a $C^{1+\delta}$ curve, Kellogg's theorem [2] tells us that $\mu(\xi) \in C^1$ and has a nonvanishing derivative in $|\xi| \geqslant 1$. We conclude, therefore, that the map of $|\xi| = e^{2c\epsilon}$ lies in R_ϵ for some small fixed $c > 0$. We also conclude from the nonvanishing of $\mu'(\xi)$ that the map of this circle lies at least $d\epsilon$ from R where d is some other fixed positive constant.

Now write

$$p(w) = \frac{1}{(2\pi i)^2}\int_{|t|=e^{-\epsilon c}}\int_{|\xi|=e^{2\epsilon c}} \frac{\mu'(\xi t)}{\mu(\xi t) - w}\,\frac{1-t^n}{1-t}\, f(\mu(\xi))\, d\xi\, dt$$

and observe first of all that this is a *polynomial* of degree $< n$ in w. We have, namely,

$$p^{(n)}(w) = \frac{n!}{(2\pi i)^2}\int_{|t|=e^{-\epsilon c}}\int_{|\xi|=e^{2c\epsilon}} \frac{\mu'(\xi t)}{(\mu(\xi t) - w)^{n+1}}\,\frac{1-t^n}{1-t}\, f(\mu(\xi))\, d\xi\, dt$$

and if we move the t-contour to a very large circle we observe that $\mu'(\xi t)/(\mu(\xi t) - w)^{n+1} \approx \rho/(\rho\xi t)^{n+1}$ while $(1 - t^n)/(1 - t) \approx t^{n-1}$. Thus we obtain an integrand which is $O(t^{-2})$ and this shows that the integral is 0.

Next observe that

$$f(w) = \frac{1}{(2\pi i)^2}\int_{|t|=e^{-c\epsilon}}\int_{|\xi|=e^{2c\epsilon}} \frac{\mu'(\xi t)}{\mu(\xi t) - w}\,\frac{1}{1-t}\, f(\mu(\xi))\, d\xi\, dt.$$

For if we again allow the t-contour to change to a large circle, we pick up a residue at $t = 1$ equal to

$$\frac{1}{2\pi i}\int_{|\xi|=e^{2c\epsilon}} \frac{\mu'(\xi)}{\mu(\xi) - w}\, f(\mu(\xi))\, d\xi = \frac{1}{2\pi i}\int_C \frac{f(\mu)}{\mu - w}\, d\mu = f(w).$$

The new integrand is like $(\rho/\rho\xi t)\cdot(-1/t)$ which is $O(t^{-2})$, and so the integral is again 0.

Thus we have

$$|f(w) - p(w)| = \left| \frac{1}{(2\pi i)^2} \int_{|t|=e^{-\epsilon c}} \int_{|\xi|=e^{2\epsilon c}} \frac{t^n}{1-t} \frac{\mu'(\xi t)}{\mu(\xi t) - w} f(\mu(\xi))\, d\xi\, dt \right|$$

$$\leqslant e^{-\epsilon n c} \frac{1}{1 - e^{-\epsilon c}} \cdot \frac{A}{d\epsilon} \cdot M$$

$$\leqslant \frac{AM}{\epsilon^2} e^{-\epsilon n c}$$

as promised. If we apply Lemma 3, to $U(T^{-1}(w))$ with b/k replacing ϵ and M given by (16), we obtain an $S(w)$ of degree $< n$ in w and $1/w$ such that on K

$$| U(T^{-1}(w)) - S(w)| \leqslant (Ak^3/b^3)e\, 2a(m/k)\, e^{-c}(b/k). \tag{17}$$

Finally, we estimate $UT(^{-1}(w(\xi)) - U(\xi)$ on C. By (5) and (14) we know that $T^{-1}(w(\xi))$ lies in $C_{Bk} - (1 + \delta)$, so that we have

$$\left| U(T^{-1}w(\xi)) - U(\xi) \right| \leqslant | T^{-1}(w(\xi)) - \xi | \max_{C_{Bk-(1+\delta)}} \left| U'(\xi) \right|.$$

By (5) and (6) this first factor is bounded by $Ak^{-(1+\delta)}$ and by Lemma 2 and (14) the second one is bounded by $A \exp(2Bmk^{-(1+\delta)})$. Together these yield

$$| U(T^{-1}(w(\xi)) - U(\xi)| \leqslant A[\exp(2Bmk^{-(1+\delta)})/k^{1+\delta}]. \tag{18}$$

The proof is now completed by choosing $m = bn/4a\lambda$ and $k = n^{1/1+\delta}$. Our estimates (15), (17), and (18) become, respectively,

$$\left| f(w) - U(\xi) \right| \leqslant A/n, \qquad \left| U(T^{-1}(w)) - S(w) \right| \leqslant An^3 \exp[(-cb/2)\, n^{\delta/1+\delta}]$$

and

$$\left| U(T^{-1}(w)) - U(\xi) \right| \leqslant A/n$$

and the theorem follows by addition since clearly $n^3 \exp[(-cb/2)\, n^{\delta/1+\delta}] = O(1/n)$.

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